Uniform Interpolation with Constructive Diamond

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Abstract. Uniform interpolation is a strong form of interpolation providing an interpretation of propositional quantifiers within a propositional logic. Pitts' seminal work establishes this property for intuitionistic propositional logic relying on a sequent calculus in which naïve backward proof-search terminates. This constructive approach has been adapted to a wide range of logics, including intuitionistic modal logics. Surprisingly, no intuitionistic modal logic with independent box and diamond has yet been shown to satisfy uniform interpolation. We fill in this gap by proving the uniform interpolation property for Constructive K (CK) and Wijesekera's K (WK). We build on Pitts' technique by exploiting existing terminating calculi for CK and WK, which we prove to eliminate cut, and formalise all our results in the proof assistant Rocq. Together, our results constitute the first positive uniform interpolation results for intuitionistic modal logics with diamond.

Keywords: Uniform Interpolation · Proof Theory · Intuitionistic Modal Logic · Rocq

1 Introduction

Intuitionistic modal logics form a rich class of logics that formalise modal reasoning over an intuitionistic propositional base. They pop up in different forms and applications and are sensitive to many design choices. When considering intuitionistic modal logics with both $\Box$ and $\Diamond$, a common feature is that $\Box$ and $\Diamond$ are not interdefinable (this in contrast to classical modal logics). One traditionally identifies two main streams to define such logics: *Intuitionistic modal logic* and *Constructive modal logic*. Intuitionistic modal logics were early investigated by, e.g., Fischer Servi [29,30], Plotkin and Stirling [44], and Ewald [25] and are motivated by their algebraic connection to classical bi-modal logics [29,54–56] and their interpretation via the standard translation into first-order intuitionistic logic [48]. The base system in this setting is Intuitionistic K, here denoted IK, as a counterpart to classical modal logic K. Constructive modal logics are mostly

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motivated by computational applications. Constructive K, here denoted CK, and its modal extensions are motivated by their Curry-Howard interpretation in type theory and their categorical semantics [5,10,11,39]. A closely related constructive modal logic is Wijesekera's system introduced to model reasoning in dynamic systems [52]. We denote its propositional part by WK.[1] Recently, new approaches have been proposed to define intuitionistic versions of classical K [8,9]. In this paper we concentrate on the well-studied constructive modal logics CK and WK.

The research on these two logics is extensive and revealed some surprises in recent years. Axiomatically, the two logics form extensions of intuitionistic propositional logic IL with the following modal rule and (subset) of modal axioms below. These axioms follow classically from the usual normality axiom $\mathrm{K}_\Box$, but are independent in an intuitionistic setting. We define $\mathsf{CK} = \mathsf{IL} + \{\mathrm{K}_\Box, \mathrm{K}_\Diamond\}$ and $\mathsf{WK} = \mathsf{CK} + \{\mathrm{N}\}$, and note that $\mathsf{CK} \subset \mathsf{WK}$.

$$\frac{\emptyset \vdash \varphi}{\Gamma \vdash \Box\varphi}\ (\mathrm{Nec}) \qquad \begin{array}{ll} \mathrm{K}_\Box & \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \\ \mathrm{K}_\Diamond & \Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi) \\ \mathrm{N} & \neg\Diamond\bot \end{array}$$

For semantic characterisations of the logics one can consult [34]. Recent studies [19–21,34] surprisingly resolved a common misbelief that the $\Diamond$-free fragments of Constructive and Intuitionistic modal logics coincide: those for CK and WK coincide with the $\Box$-only logic iK, but the one for IK is strictly stronger.

We are interested in the interpolation properties for CK and WK. Interpolation properties form a central theme in logic due to their good mathematical properties and widespread applications in computer science (see [16] for a recent overview on foundations, methods and applications of interpolation). *Craig interpolation* states that if φ entails ψ, there is a formula θ in the common vocabulary of φ and ψ such that φ entails θ and θ entails ψ. The formula θ is called the *interpolant* and explains the reason why φ entails ψ. *Uniform interpolation* is stronger, as the interpolant only depends on φ and uniformly works for any ψ in a specified vocabulary. This property makes it possible to provide an interpretation of propositional quantifiers within the propositional base [43].

Proof-theoretic methods are ideal for establishing metalogical results such as interpolation. Indeed, the Craig interpolation property has been established for WK in Wijesekera's original paper using a sequent calculus [52]. In [18], terminating sequent calculi for CK and WK were designed, but the uniform interpolation question was left open. Even though other systems are designed for CK and WK including a sequent system, a natural deduction system [10], a nested sequent calculus [7] and a focused 2-sequent calculus [42] for CK, and a tableaux calculus for the richer version CCDL of WK [53], neither Craig interpolation for CK nor uniform interpolation for CK or WK has been investigated. Most surprising, to the best of our knowledge, no other interpolation result is known

[1] Wijesekera's full system could be viewed as a more general intuitionistic modal logic than the Constructive Concurrent Dynamic Logic CCDL from [53]. Some authors also denote WK as CCDL (e.g., in [18]), although, originally, CCDL was introduced over a richer syntax including program constructors.

for *any* Constructive or Intuitionistic modal logic with independent $\Box$ and $\Diamond$ [40], also not via semantic means, except for one. We came to know via personal communication with N. Bezhanishvili that the intuitionistic modal logic MIPC [12,13,45], which can be viewed as an $\mathsf{S5}$-extension of IK, does not have the Craig interpolation property.

In this paper we focus on constructive methods towards uniform interpolation. The seminal work of Pitts provides a proof-theoretic proof of uniform interpolation for IL using a strongly terminating sequent calculus. It is now a fruitful technique for proving uniform interpolation among many types of modal logic: classical modal logics [14,15,26], intuitionistic modal logics [26,38], modal substructural logics and linear logics [4,6], and non-normal modal and conditional logics [2]. However, these studies only treat mono-modal logics (with $\Box$), but do not cover intuitionistic modal logics with independent $\Box$ and $\Diamond$.

Our Contributions. We positively answer the uniform interpolation question for CK and WK. This question is specifically posed by Dalmonte, Grellois and Olivetti in [18] in which they designed strongly terminating sequent calculi for CK and WK. Our work can be seen as a follow-up in which we (1) provide single-succedent variants of these calculi and provide termination and cut-elimination proofs, (2) provide a *constructive* proof of uniform interpolation à la Pitts for CK and WK, and (3) contribute to The Rocq Prover [49] library for logics between CK and IK [34] by formalising all our results. All results described in this paper are accompanied by a clickable symbol " " leading to their formalisation.

Our work establishes the first positive uniform interpolation result among intuitionistic modal logics with independent $\Box$ and $\Diamond$. Since uniform interpolation is stronger than Craig interpolation, we immediately obtain the Craig interpolation property for CK and WK as well. The constructive approach in Rocq provides us with a uniform interpolation calculator for WK and CK already available for some classical and intuitionistic modal logics [26,27].

2 Preliminaries

In this section we fix the syntax and axiomatic calculi for CK and WK as presented in de Groot, Shillito and Clouston's work [34]. For the mechanisation of most of the elements of this section, we refer to their paper.

Using a countably infinite set of propositional variables $\mathsf{Prop} = \{p, q, r, \dots\}$, we define the language $\mathcal{L}$ via the following grammar in BNF notation ( ):

$$\varphi ::= p \in \mathsf{Prop} \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \Box\varphi \mid \Diamond\varphi$$

We abbreviate $\neg\varphi := \varphi \to \bot$ and $\top := \neg\bot$. We use Greek lowercase letters, e.g. φ, ψ, χ and δ, to denote formulas, and Greek uppercase letters, e.g. $\Gamma, \Delta, \Phi, \Psi$, for multisets of formulas. For such a multiset Γ we define the two multisets $\Box\Gamma := \{\Box\varphi \mid \varphi \in \Gamma\}$ and $\Box^{-1}\Gamma := \{\varphi \mid \Box\varphi \in \Gamma\}$ and similarly for $\Diamond\Gamma$ and $\Diamond^{-1}\Gamma$. We also write $\Gamma \uplus \Delta$ for the *disjoint union* of multisets Γ and Δ. If Γ

Axioms

A_1	$\varphi \to (\psi \to \varphi)$	A_7	$(\varphi \wedge \psi) \to \psi$
A_2	$(\varphi \to (\psi \to \chi)) \to (\varphi \to \psi) \to (\varphi \to \chi)$	A_8	$(\varphi \to \psi) \to (\varphi \to \chi) \to (\varphi \to (\psi \wedge \chi))$
A_3	$\varphi \to (\varphi \vee \psi)$	A_9	$\bot \to \varphi$
A_4	$\psi \to (\varphi \vee \psi)$	$\mathrm{K}_\Box$	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
A_5	$(\varphi \to \chi) \to (\psi \to \chi) \to ((\varphi \vee \psi) \to \chi)$	$\mathrm{K}_\Diamond$	$\Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$
A_6	$(\varphi \wedge \psi) \to \varphi$	N	$\neg\Diamond\bot$

Rules of Inference

$$\frac{\varphi \text{ is an instance of an axiom}}{\Gamma \vdash \varphi}\ (\mathrm{Ax}) \qquad \frac{\varphi \in \Gamma}{\Gamma \vdash \varphi}\ (\mathrm{El})$$

$$\frac{\emptyset \vdash \varphi}{\Gamma \vdash \Box\varphi}\ (\mathrm{Nec}) \qquad \frac{\Gamma \vdash \varphi \quad \Gamma \vdash \varphi \to \psi}{\Gamma \vdash \psi}\ (\mathrm{MP})$$

Fig. 1. Generalised Hilbert calculi CKH ( ) and WKH ( ).

is finite, $\bigvee \Gamma$ denotes the disjunction of all formulas in Γ and $\bigwedge \Gamma$ denotes the conjunction of all formulas in Γ, with the convention that $\bigvee \emptyset = \bot$ and $\bigwedge \emptyset = \top$.

To finish with the syntax, we define a notion which we use to show termination of backward proof-search in our sequent calculi.

Definition 1 ( ). *The weight $w(\varphi)$ of a formula is defined as follows.*

$$\begin{aligned} w(\bot) = w(p) &= 1 \\ w(\psi \vee \chi) = w(\psi \to \chi) &= w(\psi) + w(\chi) + 1 \\ w(\psi \wedge \chi) &= w(\psi) + w(\chi) + 2 \\ w(\Box\psi) = w(\Diamond\psi) &= w(\psi) + 1 \end{aligned}$$

We provide generalised Hilbert calculi for CK and WK, i.e. axiomatic calculi manipulating *consecutions* of the shape $\Gamma \vdash \varphi$ where Γ is a *set* of formulas. The calculus CKH for CK extends the one for intuitionistic logic IL, with its axioms and rules, with the necessitation rule (Nec) and axioms $\mathrm{K}_\Box$ and $\mathrm{K}_\Diamond$. The calculus WKH for WK is nothing but the calculus CKH augmented with the axiom N. All the axioms and rules just mentioned are presented in Fig. 1. We write $\Gamma \vdash_{\mathsf{H}} \varphi$ if $\Gamma \vdash \varphi$ is provable in H, for $\mathsf{H} \in \{\mathsf{CKH}, \mathsf{WKH}\}$.

3 Sequent Calculi for CK and WK

In this section we introduce the sequent calculi G4CK ( ) and G4WK ( ) for CK and WK, respectively. Sequents in these calculi, which are presented in Fig. 2, are expressions of the shape $\Gamma \Rightarrow \Delta$ where the *antecedent* Γ and the *succedent* Δ are both finite multisets of formulas. Note that Γ, φ and Γ, Π stand for $\Gamma \uplus \{\varphi\}$ and $\Gamma \uplus \Pi$, respectively. While antecedents are unconstrained multisets in both calculi, succedents need to have a cardinality of *exactly one* (i.e. $|\Delta| = 1$) in G4CK, and of *at most one* (i.e. $|\Delta| \leq 1$) in G4WK. This technically makes G4CK a *single-succedent* calculus, and G4WK a calculus with either *empty* or *singleton*

$$\frac{}{\Gamma, \bot \Rightarrow \Delta}\ (\bot\mathrm{L}) \qquad \frac{}{\Gamma, p \Rightarrow p}\ (\mathrm{IdP}) \qquad \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\ (\wedge\mathrm{L}) \qquad \frac{\Gamma \Rightarrow \varphi \qquad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi}\ (\wedge\mathrm{R})$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}\ (\vee\mathrm{L}) \qquad \frac{\Gamma \Rightarrow \varphi_i}{\Gamma \Rightarrow \varphi_1 \vee \varphi_2}\ (\vee R_i)(i \in \{1,2\}) \qquad \frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi}\ (\to\mathrm{R})$$

$$\frac{\Gamma, \varphi \to (\psi \to \chi) \Rightarrow \Delta}{\Gamma, (\varphi \wedge \psi) \to \chi \Rightarrow \Delta}\ (\wedge\to\mathrm{L}) \qquad \frac{\Gamma, \varphi \to \chi, \psi \to \chi \Rightarrow \Delta}{\Gamma, (\varphi \vee \psi) \to \chi \Rightarrow \Delta}\ (\vee\to\mathrm{L})$$

$$\frac{\Gamma, p, \varphi \Rightarrow \Delta}{\Gamma, p, p \to \varphi \Rightarrow \Delta}\ (p\to\mathrm{L}) \qquad \frac{\Gamma, \psi \to \chi \Rightarrow \varphi \to \psi \qquad \Gamma, \chi \Rightarrow \Delta}{\Gamma, (\varphi \to \psi) \to \chi \Rightarrow \Delta}\ (\to\to\mathrm{L})$$

$$\frac{\Box^{-1}\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \Box\varphi}\ (\Box\mathrm{R}) \qquad \frac{\Box^{-1}\Gamma \Rightarrow \varphi \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \Box\varphi \to \psi \Rightarrow \Delta}\ (\Box\to\mathrm{L})$$

$$\frac{\Box^{-1}\Gamma, \gamma \Rightarrow \varphi \qquad \Gamma, \Diamond\gamma, \psi \Rightarrow \Delta}{\Gamma, \Diamond\gamma, \Diamond\varphi \to \psi \Rightarrow \Delta}\ (\Diamond\to\mathrm{L})$$

$$\frac{\Box^{-1}\Gamma, \varphi \Rightarrow \psi}{\Gamma, \Diamond\varphi \Rightarrow \Diamond\psi}\ (\Diamond\mathrm{L}) \qquad \frac{\Box^{-1}\Gamma, \varphi \Rightarrow \Diamond^{-1}\Delta}{\Gamma, \Diamond\varphi \Rightarrow \Delta}\ (\Diamond\mathrm{L'})$$

Fig. 2. The sequent calculi G4CK and G4WK. In the former succedents Δ are singletons of the shape $\{\delta\}$, while in the latter succedents are either singletons or the empty set. The rule $(\Diamond\mathrm{L})$ (resp. $(\Diamond\mathrm{L'})$) belongs to G4CK (resp. G4WK) exclusively.

succedents.[2] For simplicity, we call both calculi single-succedent. Beyond their succedents, these calculi differ on their rule for diamond on the left: the rule $(\Diamond\mathrm{L})$ of G4CK is replaced by the rule $(\Diamond\mathrm{L'})$ in G4WK. Note that $\Diamond^{-1}\Delta$ is non-empty only if $\Delta = \{\Diamond\delta\}$. As shown below, $(\Diamond\mathrm{L'})$ is crucial to prove N.

$$\dfrac{\dfrac{\dfrac{}{\bot \Rightarrow}\ (\bot\mathrm{L})}{\Diamond\bot \Rightarrow \bot}\ (\Diamond\mathrm{L'})}{\Rightarrow \Diamond\bot \to \bot}\ (\to\mathrm{R})$$

We write $\vdash_{\mathsf{S}} \Gamma \Rightarrow \Delta$ when $\Gamma \Rightarrow \Delta$ is provable in S, with $\mathsf{S} \in \{\mathsf{G4CK}, \mathsf{G4WK}\}$.

G4CK and G4WK are connected to multiple calculi in the literature. First, they extend with rules for modalities the strongly terminating calculus G4ip for IL, which has been invented many times [23,35,51]. Second, they are extensions of Iemhoff's single-succedent calculus for iK [37] with rules for $\Diamond$. Third, our

[2] In Rocq, instead of using multisets restricted by a cardinality of at most 1, which is cumbersome, we used the `option` type ( ).

This type takes another type as argument, in our case the type of formulas, and has two constructor: `None`, simulating $\emptyset$, and `Some(`φ`)`, simulating $\{\varphi\}$.

calculi are single-succedent versions of Dalmonte, Grellois and Olivetti's multi-succedent calculi for CK and WK (which they call CCDL) [18]. As suggested by these last authors, the adaptation of their calculi to a single-succedent setting is straightforward. The only noteworthy element in this transformation is the use of the single rule ($\Diamond$L') in G4WK for the treatment of $\Diamond$ on the left, in contrast with their counterpart multi-succedent calculus which has two.

3.1 Termination, Cut Admissibility and Equivalence

In this section we prove of both our calculi that they strongly terminate, eliminate cut, and are equivalent to their corresponding axiomatic system.

First, we prove that G4CK and G4WK *strongly terminate*. We say that a calculus strongly terminates if the process of naive backward proof-search, i.e. freely and iteratively applying rules backward on a sequent, necessarily comes to a halt. To show strong termination, it is sufficient to provide a well-founded ordering on sequents which decreases upwards in any application of any rule of our calculi. To define such an order we exploit the Dershowitz-Manna order on finite multisets [22]: a finite multiset A of elements of type T is smaller than another finite multiset B of T-elements if A is obtained from B by replacing T-elements from B by finitely many T-elements which are strictly smaller according to a well-founded order over T.

Definition 2 (Sequent ordering ,). *We define the well-founded order $\prec_f$ on formulas using their weight: $\varphi \prec_f \psi$ whenever $w(\varphi) < w(\psi)$. We generate the well-founded Dershowitz-Manna order on multisets $\prec_m$ using $\prec_f$. Finally, we write $\Gamma \Rightarrow \Delta \prec \Pi \Rightarrow \Sigma$ whenever $\Gamma \uplus \Delta \prec_m \Pi \uplus \Sigma$.*

Note that a G4WK sequent $\Gamma \Rightarrow$ with an empty succedent is compared in the order over sequents via the multiset $\Gamma \uplus \emptyset = \Gamma$. A quick inspection of the rules of our calculi shows that any premise of a rule is smaller in $\prec$ than its conclusion.

Proposition 1. *The calculi* G4CK *and* G4WK *strongly terminate.*

This directly establishes the decidability of provability in G4CK and G4WK.

Proposition 2 (,). *Provability in* G4CK *and* G4WK *is decidable.*

In Theorem 2, at the end of this section, we prove that our calculi capture CK and WK, respectively. In this light, the proposition above constitutes a *constructive and mechanised* proof of decidability for these logics. Such proofs for the two logics were already given by Dalmonte et al. [18], constructively but not mechanised, and for CK in particular by Mendler and de Paiva [41], though neither constructive nor mechanised.

Second, we aim at proving cut elimination for both calculi, building on proofs of cut admissibility via local proof transformations. To get there, we need to follow a path, first established by Dyckhoff and Negri [24], paved by a succession of technical lemmas, and leading to the admissibility of contraction. We omit the majority of these intermediate lemmas, referring to our mechanisation and Dyckhoff and Negri's work [24], except for the admissibility of some useful rules.

Lemma 1 (, , , , ,). *The following rules are admissible in* G4CK *and* G4WK.

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Pi \Rightarrow \Delta}\ (wk) \qquad \frac{}{\Gamma, \varphi \Rightarrow \varphi}\ (Id) \qquad \frac{\Gamma \Rightarrow \varphi \quad \psi, \Gamma \Rightarrow \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta}\ (\to L)$$

Next, we state the admissibility of contraction.

Proposition 3 (,). *The contraction rule is admissible in* G4CK *and* G4WK.

$$\frac{\Gamma, \psi, \psi \Rightarrow \Delta}{\Gamma, \psi \Rightarrow \Delta}\ (cntr)$$

We leverage contraction to prove the admissibility of the cut rule. Our proof goes by primary induction on the weight of the cut formula and secondary (well-founded) induction on $\prec$, a standard method for terminating calculi [32,33,46,47].

Theorem 1 (,). *The additive cut rule is admissible in* G4CK *and* G4WK.

$$\frac{\Gamma \Rightarrow \varphi \quad \varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (cut)$$

As this theorem is proved syntactically via local proof transformations, it entails that both calculi *eliminate* cut: any proof in the calculus augmented with the cut rule can be transformed into a proof without instances of cut.

Third, and finally, we show that G4CK and G4WK capture exactly the logics CK and WK. For this, we exploit both the decidability of the sequent calculi and the admissibility of cut.

Theorem 2 (, , ,). *The following equivalences hold.*

$$\begin{array}{lcl} \vdash_{\mathsf{G4CK}} \Gamma \Rightarrow \Delta & \textit{if and only if} & \Gamma \vdash_{\mathsf{CK}} \Delta \\ \vdash_{\mathsf{G4WK}} \Gamma \Rightarrow \Delta & \textit{if and only if} & \Gamma \vdash_{\mathsf{WK}} \Delta \end{array}$$

Proof We focus on G4WK, as the equivalence for G4CK is treated similarly. Note that we abused notation: Δ in $\Gamma \vdash_{\mathsf{WK}} \Delta$ denotes the *formula* φ if the succedent $\Delta = \{\varphi\}$, and $\bot$ if $\Delta = \emptyset$.

For the left to right direction it suffices to show all rules of G4WK are admissible in WK.

For the right to left direction, the pen-and-paper proof boils down to showing all axioms provable and all rules of the axiomatic system admissible in $\mathsf{G4WK}$. The admissibility of cut crucially helps in the case of (MP). In the mechanisation, we encounter the following obstacle. We are trying to prove that $\vdash_{\mathsf{G4WK}} \Gamma \Rightarrow \Delta$, a statement we mechanised in Rocq in the type `Type`, which requires the construction of an *explicit* proof. As an assumption we have $\Gamma \vdash_{\mathsf{WK}} \Delta$, a statement in the type `Prop`, which promises the opaque *existence* of a proof without giving an explicit one. We are then in a dead end: we cannot build an explicit proof for $\vdash_{\mathsf{G4WK}} \Gamma \Rightarrow \Delta$ by relying on the opaque one of $\Gamma \vdash_{\mathsf{WK}} \Delta$. To circumvent this difficulty we exploit the decidability of $\mathsf{G4WK}$: if the decision procedure outputs $\vdash_{\mathsf{G4WK}} \Gamma \Rightarrow \Delta$ then we are done, else we show that the output $\nvdash_{\mathsf{G4WK}} \Gamma \Rightarrow \Delta$ and our assumption $\Gamma \vdash_{\mathsf{WK}} \Delta$ leads to a contradiction (which does not require the construction of an explicit proof). ■

4 Uniform Interpolation

We start this section by introducing general definitions and facts about uniform interpolation.

Definition 3. *A modal logic L over the language $\mathcal{L}$ has the* uniform interpolation property *if, for every $\mathcal{L}$-formula φ and variable p, there exist $\mathcal{L}$-formulas, denoted by $\forall p\varphi$ and $\exists p\varphi$, satisfying the following three properties:*

1. p-freeness: $Vars(\exists p\varphi) \subseteq Vars(\varphi) \setminus \{p\}$ *and* $Vars(\forall p\varphi) \subseteq Vars(\varphi) \setminus \{p\}$,
2. implication: $\vdash_L \varphi \to \exists p\varphi$ *and* $\vdash_L \forall p\varphi \to \varphi$, *and*
3. uniformity: *for each formula ψ with $p \notin Vars(\psi)$:*

$$\vdash_L \varphi \to \psi \& \text{ implies } \vdash_L \exists p\varphi \to \psi,$$
$$\vdash_L \psi \to \varphi \& \text{ implies } \vdash_L \psi \to \forall p\varphi.$$

Formula $\exists p\varphi$ is called the uniform post-interpolant *of φ w.r.t. p and $\forall p\varphi$ the* uniform pre-interpolant *of φ w.r.t. p.*

The notation of $\exists p\varphi$ and $\forall p\varphi$ is suggestive: the formulas do not really contain quantifiers, but they are defined over the modal language $\mathcal{L}$. This notation is justified by Pitts' result stating that uniform interpolants provide *an interpretation* for propositional quantifiers of second order intuitionistic logic into the propositional language [43].

Remark 1. It is interesting to mention that both in classical and intuitionistic based (modal) logics, the formulas $\forall p(\varphi \to \psi)$ and $\exists p(\varphi) \to \forall p(\varphi \to \psi)$ are equivalent. This was shown in [26, Lemma 1] for modal logics based on $\Box$. Since the proof only relies on intuitionistic propositional reasoning, the same result holds in the intuitionistic modal setting with $\Box$ and $\Diamond$. So in case uniform interpolants exist for CK (which we will show in this paper), we have $\vdash_{\mathsf{CK}} \forall p(\varphi \to \psi)$ if and only if $\vdash_{\mathsf{CK}} \exists p(\varphi) \to \forall p(\varphi \to \psi)$. The analogous result holds for WK.

Remark 2. In a classical setting, $\forall p\varphi$ and $\exists p\varphi$ are dual to each other, but that is not true when working in an intuitionistic logic like CK. However, note that it is possible to define formulas of the form $\exists p\varphi$ in terms of $\forall$ and $\rightarrow$ as follows using an extra propositional variable q not free in φ:

$$\exists p\varphi := \forall q(\forall p(\varphi \rightarrow q) \rightarrow q).$$

This is a folklore result when the quantifiers denote the quantifiers in second order intuitionistic propositional logic (see [43]). Here we write the quantifiers to denote uniform interpolants for which the equivalence also holds ([31], Remark 2.2.7). This means that from a method constructing all formulas of the form $\forall p\varphi$, one gets a definition of $\exists p\varphi$. However, the known proof-theoretic method [43] for IL constructs $\forall p\varphi$ and $\exists p\varphi$ via a mutual recursion.

We provide a proof-theoretic construction to prove the uniform interpolation property. In light of Remark 2 we use a sequent-style definition of uniform interpolation in which pre- and post-uniform interpolants are constructed simultaneously.

Definition 4. *A set of provable single-succedent sequents, denoted* $\vdash$*, has the* uniform interpolation property *if, for any single-succedent sequent* $\Gamma \Rightarrow \Delta$ *and variable* p*, there exist modal formulas* $\mathsf{E}_p(\Gamma)$ *and* $\mathsf{A}_p(\Gamma \Rightarrow \Delta)$ *such that the following three properties hold:*

1. p-freeness:
   (a) $\mathit{Vars}(\mathsf{E}_p(\Gamma)) \subseteq \mathit{Vars}(\Gamma) \setminus \{p\}$, *and*
   (b) $\mathit{Vars}(\mathsf{A}_p(\Gamma \Rightarrow \Delta)) \subseteq \mathit{Vars}(\Gamma, \Delta) \setminus \{p\}$;
2. implication:
   (a) $\vdash \Gamma \Rightarrow \mathsf{E}_p(\Gamma)$, *and*
   (b) $\vdash \Gamma, \mathsf{A}_p(\Gamma \Rightarrow \Delta) \Rightarrow \Delta$;
3. uniformity: *for any finite multiset of formulas* Π *such that* $p \notin \mathit{Vars}(\Pi)$*, if it holds that* $\vdash \Pi, \Gamma \Rightarrow \Delta$*, then it also holds that:*
   (a) $\vdash \Pi, \mathsf{E}_p(\Gamma) \Rightarrow \Delta$ *if* $p \notin \mathit{Vars}(\Delta)$*, and*
   (b) $\vdash \Pi, \mathsf{E}_p(\Gamma) \Rightarrow \mathsf{A}_p(\Gamma \Rightarrow \Delta)$.

We say that a sequent calculus S *has the* uniform interpolation property *if* $\vdash_{\mathsf{S}}$ *has the uniform interpolation property.*

Note that in the above definition, we have $|\Delta| = 1$ for $\mathsf{G4CK}$ and $|\Delta| \leq 1$ for $\mathsf{G4WK}$. In the latter case, Definition 4 is equivalent to the uniform interpolant properties from [38]. We have the following well-known fact.

Lemma 2. *Suppose sequent calculus* S *is sound and complete with respect to logic* L*. If* S *has the uniform interpolation property, then* L *has the uniform interpolation property.*

Proof. The proof is well-known and can be found in [14,43]. The essence is to define $\exists p\varphi := \mathsf{E}_p(\varphi)$ and $\forall p\varphi := \mathsf{A}_p(\ \Rightarrow \varphi)$. ■

4.1 Uniform Interpolation for CK

To prove the uniform interpolation property for CK, we use the sequent calculus $\mathsf{G4CK}$ to construct $\mathsf{E}_p(\Gamma)$ and $\mathsf{A}_p(s)$ for any multiset Γ and any sequent s. Our construction adapts the one for iK [38] by adding extra cases for the $\Diamond$.

Definition 5. *We define sets $\mathcal{E}_\mathsf{p}(\Gamma)$ and $\mathcal{A}_\mathsf{p}(s)$ based on a mutual recursion on the ordering $\prec$ according to the rows in Fig. 3. We define :*

$$\mathsf{E}_p(\Gamma) := \bigwedge \mathcal{E}_\mathsf{p}(\Gamma) \text{ and } \mathsf{A}_p(s) := \bigvee \mathcal{A}_\mathsf{p}(s).$$

Theorem 3. *The algorithm defining $\mathsf{E}_p(\Gamma)$ and $\mathsf{A}_p(\Gamma \Rightarrow \varphi)$ is terminating.*

Proof. $\mathsf{E}_p(\Gamma)$ and $\mathsf{A}_p(\Gamma \Rightarrow \varphi)$ are defined by mutual induction on the ordering $\prec$ where for $\mathsf{A}_p(\Gamma \Rightarrow \varphi)$ we look at the multiset Γ, φ. Each recursive call in Fig. 3 reduces in that ordering. Moreover, for each multiset Γ there are finitely many matches of rows ($\mathsf{E}_p^{\mathsf{CK}}0$)-($\mathsf{E}_p^{\mathsf{CK}}13$), and similarly so for sequent $\Gamma \Rightarrow \varphi$ and rows ($\mathsf{A}_p^{\mathsf{CK}}1'$)-($\mathsf{A}_p^{\mathsf{CK}}19$). This means that $\mathcal{A}_\mathsf{p}(\Gamma \Rightarrow \varphi)$ and $\mathcal{E}_\mathsf{p}(\Gamma)$ are finite sets of formulas. So, the constructions of $\mathsf{E}_p(\Gamma)$ and $\mathsf{A}_p(\Gamma \Rightarrow \varphi)$ are terminating. ∎

Let us explain the construction of Fig. 3. Rows ($\mathsf{E}_p^{\mathsf{CK}}0$), ($\mathsf{E}_p^{\mathsf{CK}}1'$), ($\mathsf{E}_p^{\mathsf{CK}}4'$) and ($\mathsf{A}_p^{\mathsf{CK}}1'$), ($\mathsf{A}_p^{\mathsf{CK}}4'$), ($\mathsf{A}_p^{\mathsf{CK}}9$) take care of boolean constants and variables other than p, which can be smartly taken out of the recursive call since these are atomic p-free formulas. Rows ($\mathsf{E}_p^{\mathsf{CK}}1''$), ($\mathsf{E}_p^{\mathsf{CK}}4''$) and rows ($\mathsf{A}_p^{\mathsf{CK}}1''$), ($\mathsf{A}_p^{\mathsf{CK}}4''$) are not necessary for the construction, as $\bot$ is a neutral element for $\mathsf{E}_p(\cdot)$ and $\top$ is a neutral element for $\mathsf{A}_p(\cdot)$, but are required in Rocq to ensure the functionality of $\mathsf{E}_p(\cdot)$ and $\mathsf{A}_p(\cdot)$ by making them defined on all inputs.

Rows ($\mathsf{E}_p^{\mathsf{CK}}2$), ($\mathsf{E}_p^{\mathsf{CK}}3$), ($\mathsf{E}_p^{\mathsf{CK}}5'$)-($\mathsf{E}_p^{\mathsf{CK}}8'$), ($\mathsf{E}_p^{\mathsf{CK}}10$), ($\mathsf{E}_p^{\mathsf{CK}}12$) and ($\mathsf{A}_p^{\mathsf{CK}}2$), ($\mathsf{A}_p^{\mathsf{CK}}3$), ($\mathsf{A}_p^{\mathsf{CK}}5'$)-($\mathsf{A}_p^{\mathsf{CK}}8'$), ($\mathsf{A}_p^{\mathsf{CK}}11$)-($\mathsf{A}_p^{\mathsf{CK}}17$) correspond to what we call a *full* application of a calculus rule where the middle column presents the conclusion of the rule and the right column uses the premise of that rule in the recursive call.

Rows ($\mathsf{E}_p^{\mathsf{CK}}9$), ($\mathsf{E}_p^{\mathsf{CK}}11$), ($\mathsf{E}_p^{\mathsf{CK}}13$), ($\mathsf{A}_p^{\mathsf{CK}}18$) and ($\mathsf{A}_p^{\mathsf{CK}}19$) correspond to what we call *partial* applications of a rule from the calculus. When considering the *uniformity* property of uniform interpolation in Definition 4, a rule application might be possible due to extra formulas in the p-free context Π outside the reach of the interpolant construction meaning that these rows in the table *partially* match the form of rule from the calculus. For example, row ($\mathsf{E}_p^{\mathsf{CK}}11$) is used in case there is a formula $\Diamond\delta_1' \to \delta_2'$ in the hidden p-free context that would together with $\Diamond\delta \in \Gamma$ make an application of the rule ($\Diamond\!\to$L) possible. Row ($\mathsf{E}_p^{\mathsf{CK}}9$) is used for partial applications of ($\Box$R) and ($\Diamond$L).

We will use the following simple facts.

Lemma 3.

1. *Let Γ be a multiset such that $\mathcal{E}_\mathsf{p}(\Gamma) \neq \emptyset$. For any $\chi \in \mathcal{E}_\mathsf{p}(\Gamma)$, multiset Π and formula φ, if $\vdash_{\mathsf{G4CK}} \Pi, \chi \Rightarrow \varphi$, then $\vdash_{\mathsf{G4CK}} \Pi, \mathsf{E}_p(\Gamma) \Rightarrow \varphi$.*

	Γ matches	$\mathcal{E}_{\mathsf{p}}(\Gamma)$ contains
$(\mathsf{E}_p^{\mathsf{CK}}0)$	$\Gamma', \bot$	$\bot$
$(\mathsf{E}_p^{\mathsf{CK}}1')$	Γ', q	q
$(\mathsf{E}_p^{\mathsf{CK}}1'')$	Γ', p	$\top$
$(\mathsf{E}_p^{\mathsf{CK}}2)$	$\Gamma', \psi_1 \wedge \psi_2$	$\mathsf{E}_p(\Gamma', \psi_1, \psi_2)$
$(\mathsf{E}_p^{\mathsf{CK}}3)$	$\Gamma', \psi_1 \vee \psi_2$	$\mathsf{E}_p(\Gamma', \psi_1) \vee \mathsf{E}_p(\Gamma', \psi_2)$
$(\mathsf{E}_p^{\mathsf{CK}}4')$	$\Gamma', q \to \psi$ with $q \notin \Gamma'$	$q \to \mathsf{E}_p(\Gamma', \psi)$
$(\mathsf{E}_p^{\mathsf{CK}}4'')$	$\Gamma', p \to \psi$ with $p \notin \Gamma'$	$\top$
$(\mathsf{E}_p^{\mathsf{CK}}5')$	$\Gamma', r, r \to \psi$	$\mathsf{E}_p(\Gamma', r, \psi)$
$(\mathsf{E}_p^{\mathsf{CK}}6)$	$\Gamma', (\delta_1 \wedge \delta_2) \to \delta_3$	$\mathsf{E}_p(\Gamma', \delta_1 \to (\delta_2 \to \delta_3))$
$(\mathsf{E}_p^{\mathsf{CK}}7)$	$\Gamma', (\delta_1 \vee \delta_2) \to \delta_3$	$\mathsf{E}_p(\Gamma', \delta_1 \to \delta_3, \delta_2 \to \delta_3)$
$(\mathsf{E}_p^{\mathsf{CK}}8')$	$\Gamma', (\delta_1 \to \delta_2) \to \delta_3$	$[\mathsf{E}_p(\Gamma', \delta_2 \to \delta_3, \delta_1) \to \mathsf{A}_p(\Gamma', \delta_2 \to \delta_3, \delta_1 \Rightarrow \delta_2)]$ $\to \mathsf{E}_p(\Gamma', \delta_3)$
$(\mathsf{E}_p^{\mathsf{CK}}9)$	$\Gamma \neq \emptyset$	$\Box \mathsf{E}_p(\Box^{-1}\Gamma)$
$(\mathsf{E}_p^{\mathsf{CK}}10)$	$\Gamma', \Box\delta_1 \to \delta_2$	$\Box[\mathsf{E}_p(\Box^{-1}\Gamma') \to \mathsf{A}_p(\Box^{-1}\Gamma' \Rightarrow \delta_1)] \to \mathsf{E}_p(\Gamma', \delta_2)$
$(\mathsf{E}_p^{\mathsf{CK}}11)$	$\Gamma', \Diamond\delta$	$\Diamond \mathsf{E}_p(\Box^{-1}\Gamma', \delta)$
$(\mathsf{E}_p^{\mathsf{CK}}12)$	$\Gamma', \Diamond\delta_3, \Diamond\delta_1 \to \delta_2$	$\Box[\mathsf{E}_p(\Box^{-1}\Gamma', \delta_3) \to \mathsf{A}_p(\Box^{-1}\Gamma', \delta_3 \Rightarrow \delta_1)]$ $\to \mathsf{E}_p(\Gamma', \Diamond\delta_3, \delta_2)$
$(\mathsf{E}_p^{\mathsf{CK}}13)$	$\Gamma', \Diamond\delta_1 \to \delta_2$	$\Diamond[\mathsf{E}_p(\Box^{-1}\Gamma') \to \mathsf{A}_p(\Box^{-1}\Gamma' \Rightarrow \delta_1)] \to \mathsf{E}_p(\Gamma', \delta_2)$

	s matches	$\mathcal{A}_{\mathsf{p}}(s)$ contains
$(\mathsf{A}_p^{\mathsf{CK}}1')$	$\Gamma, q \Rightarrow \varphi$	$\bot$
$(\mathsf{A}_p^{\mathsf{CK}}1'')$	$\Gamma, p \Rightarrow \varphi$ with $\varphi \neq p$	$\bot$
$(\mathsf{A}_p^{\mathsf{CK}}2)$	$\Gamma, \psi_1 \wedge \psi_2 \Rightarrow \varphi$	$\mathsf{A}_p(\Gamma, \psi_1, \psi_2 \Rightarrow \varphi)$
$(\mathsf{A}_p^{\mathsf{CK}}3)$	$\Gamma, \psi_1 \vee \psi_2 \Rightarrow \varphi$	$[\mathsf{E}_p(\Gamma, \psi_1) \to \mathsf{A}_p(\Gamma, \psi_1 \Rightarrow \varphi)] \wedge$ $[\mathsf{E}_p(\Gamma, \psi_2) \to \mathsf{A}_p(\Gamma, \psi_2 \Rightarrow \varphi)]$
$(\mathsf{A}_p^{\mathsf{CK}}4')$	$\Gamma, q \to \psi \Rightarrow \varphi$ with $q \notin \Gamma$	$q \wedge \mathsf{A}_p(\Gamma, \psi \Rightarrow \varphi)$
$(\mathsf{A}_p^{\mathsf{CK}}4'')$	$\Gamma, p \to \psi \Rightarrow \varphi$ with $p \notin \Gamma$	$\bot$
$(\mathsf{A}_p^{\mathsf{CK}}5')$	$\Gamma, r, r \to \psi \Rightarrow \varphi$	$\mathsf{A}_p(\Gamma, r, \psi \Rightarrow \varphi)$
$(\mathsf{A}_p^{\mathsf{CK}}6)$	$\Gamma, (\delta_1 \wedge \delta_2) \to \delta_3 \Rightarrow \varphi$	$\mathsf{A}_p(\Gamma, \delta_1 \to (\delta_2 \to \delta_3) \Rightarrow \varphi)$
$(\mathsf{A}_p^{\mathsf{CK}}7)$	$\Gamma, (\delta_1 \vee \delta_2) \to \delta_3 \Rightarrow \varphi$	$\mathsf{A}_p(\Gamma, \delta_1 \to \delta_3, \delta_2 \to \delta_3 \Rightarrow \varphi)$
$(\mathsf{A}_p^{\mathsf{CK}}8')$	$\Gamma, (\delta_1 \to \delta_2) \to \delta_3 \Rightarrow \varphi$	$[\mathsf{E}_p(\Gamma, \delta_2 \to \delta_3, \delta_1) \to \mathsf{A}_p(\Gamma, \delta_2 \to \delta_3, \delta_1 \Rightarrow \delta_2)]$ $\wedge\ \mathsf{A}_p(\Gamma, \delta_3 \Rightarrow \varphi)$
$(\mathsf{A}_p^{\mathsf{CK}}9)$	$\Gamma \Rightarrow q$	q
$(\mathsf{A}_p^{\mathsf{CK}}10)$	$\Gamma, p \Rightarrow p$	$\top$
$(\mathsf{A}_p^{\mathsf{CK}}11)$	$\Gamma \Rightarrow \varphi_1 \wedge \varphi_2$	$\mathsf{A}_p(\Gamma \Rightarrow \varphi_1) \wedge \mathsf{A}_p(\Gamma \Rightarrow \varphi_2)$
$(\mathsf{A}_p^{\mathsf{CK}}12)$	$\Gamma \Rightarrow \varphi_1 \vee \varphi_2$	$\mathsf{A}_p(\Gamma \Rightarrow \varphi_1) \vee \mathsf{A}_p(\Gamma \Rightarrow \varphi_2)$
$(\mathsf{A}_p^{\mathsf{CK}}13)$	$\Gamma \Rightarrow \varphi_1 \to \varphi_2$	$\mathsf{E}_p(\Gamma, \varphi_1) \to \mathsf{A}_p(\Gamma, \varphi_1 \Rightarrow \varphi_2)$
$(\mathsf{A}_p^{\mathsf{CK}}14)$	$\Gamma \Rightarrow \Box\delta$	$\Box(\mathsf{E}_p(\Box^{-1}\Gamma) \to \mathsf{A}_p(\Box^{-1}\Gamma \Rightarrow \delta))$
$(\mathsf{A}_p^{\mathsf{CK}}15)$	$\Gamma, \Box\delta_1 \to \delta_2 \Rightarrow \varphi$	$\Box[\mathsf{E}_p(\Box^{-1}\Gamma) \to \mathsf{A}_p(\Box^{-1}\Gamma \Rightarrow \delta_1)] \wedge \mathsf{A}_p(\Gamma, \delta_2 \Rightarrow \varphi)$
$(\mathsf{A}_p^{\mathsf{CK}}16)$	$\Gamma, \Diamond\delta_1 \Rightarrow \Diamond\delta_2$	$\Box(\mathsf{E}_p(\Box^{-1}\Gamma, \delta_1) \to \mathsf{A}_p(\Box^{-1}\Gamma, \delta_1 \Rightarrow \delta_2))$
$(\mathsf{A}_p^{\mathsf{CK}}17)$	$\Gamma, \Diamond\delta_3, \Diamond\delta_1 \to \delta_2 \Rightarrow \varphi$	$\Box[\mathsf{E}_p(\Box^{-1}\Gamma, \delta_3) \to \mathsf{A}_p(\Box^{-1}\Gamma, \delta_3 \Rightarrow \delta_1)]$ $\wedge \mathsf{A}_p(\Gamma, \Diamond\delta_3, \delta_2 \Rightarrow \varphi)$
$(\mathsf{A}_p^{\mathsf{CK}}18)$	$\Gamma \Rightarrow \Diamond\delta$	$\Diamond(\mathsf{E}_p(\Box^{-1}\Gamma) \to \mathsf{A}_p(\Box^{-1}\Gamma \Rightarrow \delta))$
$(\mathsf{A}_p^{\mathsf{CK}}19)$	$\Gamma, \Diamond\delta_1 \to \delta_2 \Rightarrow \varphi$	$\Diamond[\mathsf{E}_p(\Box^{-1}\Gamma) \to \mathsf{A}_p(\Box^{-1}\Gamma \Rightarrow \delta_1)] \wedge \mathsf{A}_p(\Gamma, \delta_2 \Rightarrow \varphi)$

Fig. 3. Constructions of $\mathsf{E}_p(\Gamma)$ and $\mathsf{A}_p(\Gamma \Rightarrow \varphi)$ for $\mathsf{G4CK}$ where in all clauses $q \neq p$. The constructions are divided into three parts: rows for propositional logic IL that are a slight modification from [43] (indicated here with $'$ or $''$), rows that cover the box modality as in [38] for box-only logic iK, and rows that cover the diamond.

2. *Let $\Gamma \Rightarrow \varphi$ be a sequent such that $\mathcal{A}_\mathsf{p}(\Gamma \Rightarrow \varphi) \neq \emptyset$. For any $\chi \in \mathcal{A}_\mathsf{p}(\Gamma \Rightarrow \varphi)$ and multiset Π, if $\vdash_{\mathsf{G4CK}} \Pi \Rightarrow \chi$, then $\vdash_{\mathsf{G4CK}} \Pi \Rightarrow \mathsf{A}_p(\Gamma \Rightarrow \varphi)$.* (, , ,)

Proof. (1) follows from the fact that $\mathsf{E}_p(\Gamma) = \bigwedge \mathcal{E}_\mathsf{p}(\Gamma)$. So multiple applications of weakening and $(\wedge\mathrm{L})$ applied to $\Pi, \chi \Rightarrow \varphi$ gives a proof for $\Pi, \mathsf{E}_p(\Gamma) \Rightarrow \varphi$. For (2), observe that $\mathsf{A}_p(\Gamma \Rightarrow \varphi) = \bigvee \mathcal{A}_\mathsf{p}(\Gamma \Rightarrow \varphi)$, so multiple applications of $(\vee\mathrm{R}_1)$ and $(\vee\mathrm{R}_2)$ applied to $\Pi \Rightarrow \chi$ provide a proof for $\Pi \Rightarrow \mathsf{A}_p(\Gamma \Rightarrow \varphi)$. ■

Lemma 4 ( ). *If $\vdash_{\mathsf{G4CK}} \Pi, \Box\mathsf{E}_p(\Box^{-1}\Gamma) \Rightarrow \varphi$, then $\vdash_{\mathsf{G4CK}} \Pi, \mathsf{E}_p(\Gamma) \Rightarrow \varphi$.*

Proof If $\Gamma = \emptyset$, then $\Box^{-1}\Gamma = \emptyset$ and so $\Box\mathsf{E}_p(\Box^{-1}\Gamma) = \Box\top = \top$. In that case $\vdash_{\mathsf{G4CK}} \Pi, \top \Rightarrow \varphi$, and so $\vdash_{\mathsf{G4CK}} \Pi, \mathsf{E}_p(\Gamma) \Rightarrow \varphi$ by weakening. If $\Gamma \neq \emptyset$ we know by $(\mathsf{E}_p^{\mathsf{CK}}9)$ that $\Box\mathsf{E}_p(\Box^{-1}\Gamma) \in \mathcal{E}_\mathsf{p}(\Gamma)$. So by Lemma 3 we conclude that $\vdash_{\mathsf{G4CK}} \Pi, \mathsf{E}_p(\Gamma) \Rightarrow \varphi$. ■

We now present the main theorem of this section. Due to space restrictions we highlight some of the important steps of the proof below.

Theorem 4. *Sequent calculus* G4CK *has the uniform interpolation property.*

Proof. We take the construction of $\mathsf{A}_p(\cdot)$ and $\mathsf{E}_p(\cdot)$ from Definition 5. We have to show all properties from Definition 4. The *p-freeness* property follows easily by examining the construction of $\mathsf{A}_p(\cdot)$ and $\mathsf{E}_p(\cdot)$ ( ).

The *implication* properties (2a) and (2b) are simultaneously proved by the multiset ordering $\prec$ on Γ, φ, i.e., we prove (2a) $\vdash_{\mathsf{G4CK}} \Gamma \Rightarrow \mathsf{E}_p(\Gamma)$, and (2b) $\vdash_{\mathsf{G4CK}} \Gamma, \mathsf{A}_p(\Gamma \Rightarrow \varphi) \Rightarrow \varphi$. The implementation ( ) is an elegant extension of the existing implementation for IL [27]. For (2a), we have $\mathsf{E}_p(\Gamma) := \bigwedge \mathcal{E}_\mathsf{p}(\Gamma)$ given in Fig. 3, so it is sufficient to prove every conjunct in $\mathcal{E}_\mathsf{p}(\Gamma)$. Let us here treat two cases for $\Diamond$.

Case $(\mathsf{E}_p^{\mathsf{CK}}9)$:
In this case $\Gamma \neq \emptyset$, which means that $\Box^{-1}\Gamma \prec \Gamma$, so the induction hypothesis (IH) applies:

$$\dfrac{\overset{\text{(IH)}}{\Box^{-1}\Gamma \Rightarrow \mathsf{E}_p(\Box^{-1}\Gamma)}}{\Gamma \Rightarrow \Box\mathsf{E}_p(\Box^{-1}\Gamma)}\ (\Box\mathrm{R})$$

Case $(\mathsf{E}_p^{\mathsf{CK}}13)$:
Let us write $\chi := \mathsf{E}_p(\Box^{-1}\Gamma') \to \mathsf{A}_p(\Box^{-1}\Gamma' \Rightarrow \delta_1)$. We have the following derivation in which we use the rules $(\to\mathrm{L})$ and (wk) shown admissible in Lemma 1.

$$\dfrac{\dfrac{\dfrac{\overset{\text{(IH)}}{\Box^{-1}\Gamma' \Rightarrow \mathsf{E}_p(\Box^{-1}\Gamma')} \quad \overset{\text{(IH)}}{\Box^{-1}\Gamma', \mathsf{A}_p(\Box^{-1}\Gamma' \Rightarrow \delta_1) \Rightarrow \delta_1}}{\Box^{-1}\Gamma', \chi \Rightarrow \delta_1}\ (\to\mathrm{L}) \qquad \dfrac{\overset{\text{(IH)}}{\Gamma', \delta_2 \Rightarrow \mathsf{E}_p(\Gamma', \delta_2)}}{\Gamma', \delta_2, \Diamond\chi \Rightarrow \mathsf{E}_p(\Gamma', \delta_2)}\ (\mathrm{wk})}{\Gamma', \Diamond\delta_1 \to \delta_2, \Diamond\chi \Rightarrow \mathsf{E}_p(\Gamma', \delta_2)}\ (\Diamond\!\to\!\mathrm{L})}{\Gamma', \Diamond\delta_1 \to \delta_2 \Rightarrow \Diamond\chi \to \mathsf{E}_p(\Gamma', \delta_2)}\ (\to\mathrm{R})$$

For (2b) we prove that $\vdash_{\mathsf{G4CK}} \Gamma, \mathsf{A}_p(\Gamma \Rightarrow \varphi) \Rightarrow \varphi$. We have $\mathsf{A}_p(\Gamma \Rightarrow \varphi) := \bigvee \mathcal{A}_{\mathsf{p}}(\Gamma \Rightarrow \varphi)$ given in Fig. 3, so it is sufficient to prove every disjunct in $\mathsf{A}_p(\Gamma \Rightarrow \varphi)$ placed on the left of the sequent. As an example, we treat the following case.

Case ($\mathsf{A}_p^{\mathsf{CK}}19$):
Let us write $\chi := \mathsf{E}_p(\Box^{-1}\Gamma') \to \mathsf{A}_p(\Box^{-1}\Gamma' \Rightarrow \delta_1)$. Note that in the following derivation, $\Box^{-1}\mathsf{A}_p(\Gamma', \delta_2 \Rightarrow \varphi)$ represents a formula when $\mathsf{A}_p(\Gamma', \delta_2 \Rightarrow \varphi)$ is a boxed formula and $\emptyset$ when $\mathsf{A}_p(\Gamma', \delta_2 \Rightarrow \varphi)$ is not a boxed formula. In both cases, our weakening rule from Lemma 1 works.

$$
\dfrac{
\dfrac{
\dfrac{
\overset{\text{(IH)}}{\Box^{-1}\Gamma' \Rightarrow \mathsf{E}_p(\Box^{-1}\Gamma')}
\qquad
\overset{\text{(IH)}}{\Box^{-1}\Gamma', \mathsf{A}_p(\Box^{-1}\Gamma' \Rightarrow \delta_1) \Rightarrow \delta_1}
}{\Box^{-1}\Gamma', \chi \Rightarrow \delta_1}\,(\to\text{L})
}{\Box^{-1}\Gamma', \chi, \Box^{-1}\mathsf{A}_p(\Gamma', \delta_2 \Rightarrow \varphi) \Rightarrow \delta_1}\,(\text{wk})
\qquad
\dfrac{\overset{\text{(IH)}}{\Gamma', \delta_2, \mathsf{A}_p(\Gamma', \delta_2 \Rightarrow \varphi) \Rightarrow \varphi}}{\Gamma', \delta_2, \Diamond\chi, \mathsf{A}_p(\Gamma', \delta_2 \Rightarrow \varphi) \Rightarrow \varphi}\,(\text{wk})
}{
\dfrac{\Gamma', \Diamond\delta_1 \to \delta_2, \Diamond\chi, \mathsf{A}_p(\Gamma', \delta_2 \Rightarrow \varphi) \Rightarrow \varphi}{\Gamma', \Diamond\delta_1 \to \delta_2, \Diamond\chi \wedge \mathsf{A}_p(\Gamma', \delta_2 \Rightarrow \varphi) \Rightarrow \varphi}\,(\wedge\text{L})
}\,(\Diamond\!\to\!\text{L})
$$

Now we turn to the *uniformity* property of uniform interpolation ( ). Let Π be a finite multiset such that $p \notin \mathrm{Vars}(\Pi)$ and suppose $\vdash_{\mathsf{G4CK}} \Pi, \Gamma \Rightarrow \varphi$ by proof π. By induction on $\Pi, \Gamma \Rightarrow \varphi$ via $\prec$ we simultaneously prove that (3a) $\vdash \Pi, \mathsf{E}_p(\Gamma) \Rightarrow \varphi$ if $p \notin \mathrm{Vars}(\varphi)$ and (3b) $\vdash_{\mathsf{G4CK}} \Pi, \mathsf{E}_p(\Gamma) \Rightarrow \mathsf{A}_p(\Gamma \Rightarrow \varphi)$. In the sequel we use primes to denote that a formula is p-free, so for (3a) we write $\varphi = \varphi'$. We look at the last rule applied in proof π. Here we only discuss the rule $(\Diamond\text{L})$.

Case $(\Diamond\text{L})$:
Subcase: The principal formula is in p-free Π, i.e., $\Pi = \Pi_0, \Diamond\delta_1'$.

(a) The conclusion of the rule is of the form $\Pi_0, \Diamond\delta_1', \Gamma \Rightarrow \Diamond\varphi_0'$ with p-free formula φ_0'. By induction we have $\vdash_{\mathsf{G4CK}} \Box^{-1}\Pi_0, \delta_1', \mathsf{E}_p(\Box^{-1}\Gamma) \Rightarrow \varphi_0'$. We obtain the desired result with the following derivation:

$$
\dfrac{\dfrac{\overset{\text{(IH)}}{\Box^{-1}\Pi_0, \delta_1', \mathsf{E}_p(\Box^{-1}\Gamma) \Rightarrow \varphi_0'}}{\Pi_0, \Diamond\delta_1', \Box\mathsf{E}_p(\Box^{-1}\Gamma) \Rightarrow \Diamond\varphi_0'}\,(\Diamond\text{L})}{\Pi_0, \Diamond\delta_1', \mathsf{E}_p(\Gamma) \Rightarrow \Diamond\varphi_0'}\,\text{Lemma } 4
$$

(b) The conclusion of the rule is of the form $\Pi_0, \Diamond\delta_1', \Gamma \Rightarrow \Diamond\varphi$. By induction, we get $\vdash_{\mathsf{G4CK}} \Box^{-1}\Pi_0, \delta_1', \mathsf{E}_p(\Box^{-1}\Gamma) \Rightarrow \mathsf{A}_p(\Box^{-1}\Gamma \Rightarrow \varphi)$. We have the following:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{(IH)}}{\Box^{-1}\Pi_0, \delta_1', \mathsf{E}_p(\Box^{-1}\Gamma) \Rightarrow \mathsf{A}_p(\Box^{-1}\Gamma \Rightarrow \varphi)}}{\Box^{-1}\Pi_0, \delta_1' \Rightarrow \mathsf{E}_p(\Box^{-1}\Gamma) \to \mathsf{A}_p(\Box^{-1}\Gamma \Rightarrow \varphi)}\,(\to\text{R})}{\Pi_0, \Diamond\delta_1' \Rightarrow \Diamond[\mathsf{E}_p(\Box^{-1}\Gamma) \to \mathsf{A}_p(\Box^{-1}\Gamma \Rightarrow \varphi)]}\,(\Diamond\text{L})}{\Pi_0, \Diamond\delta_1', \mathsf{E}_p(\Gamma) \Rightarrow \Diamond[\mathsf{E}_p(\Box^{-1}\Gamma) \to \mathsf{A}_p(\Box^{-1}\Gamma \Rightarrow \varphi)]}\,(\text{wk})}{\Pi_0, \Diamond\delta_1', \mathsf{E}_p(\Gamma) \Rightarrow \mathsf{A}_p(\Gamma \Rightarrow \Diamond\varphi)}\,(\mathsf{A}_p^{\mathsf{CK}}18), \text{ Lemma } 3
$$

Subcase: The principal formula is in Γ, i.e., $\Gamma = \Gamma_0, \Diamond\delta_1$.

(a) This case proceeds in a similar way as the previous subcase (a), where now, instead of Lemma 4 we use Lemma 3 with ($\mathsf{E}_p^{\mathsf{CK}}11$).
(b) This case proceeds in a similar way as the previous subcase (b), where, instead of ($\mathsf{A}_p^{\mathsf{CK}}18$) we use ($\mathsf{A}_p^{\mathsf{CK}}16$) and rule (□R) instead of (◇L).

We have checked the properties of p-freeness, implication, and uniformity for sequent calculus $\mathsf{G4CK}$, concluding that $\mathsf{G4CK}$ has uniform interpolation. ■

Corollary 1 ( ). *Logic* CK *has the uniform interpolation property.*

4.2 Uniform Interpolation for WK

The construction of uniform interpolants for WK only requires two modifications compared to the construction of uniform interpolants for CK in Fig. 3, because the two calculi $\mathsf{G4CK}$ and $\mathsf{G4WK}$ only differ in two aspects. First, since sequents $\Gamma \Rightarrow \Delta$ in $\mathsf{G4WK}$ obey $|\Delta| \leq 1$ instead of $|\Delta| = 1$, we should read any row in Fig. 3 that matches a sequent s with φ in the succedent, as having Δ in the succedent with $|\Delta| \leq 1$. So we replace all occurrences of φ (without a subscript) in Fig. 3 with Δ. To be precise, this applies to rows ($\mathsf{A}_p^{\mathsf{CK}}1'$)-($\mathsf{A}_p^{\mathsf{CK}}8'$), ($\mathsf{A}_p^{\mathsf{CK}}15$), ($\mathsf{A}_p^{\mathsf{CK}}17$), and ($\mathsf{A}_p^{\mathsf{CK}}19$).

Second, the main difference between the calculi $\mathsf{G4CK}$ and $\mathsf{G4WK}$ lies in the left rule for ◇ which in $\mathsf{G4WK}$ has the form:

$$\frac{\Box^{-1}\Gamma, \varphi \Rightarrow \Diamond^{-1}\Delta}{\Gamma, \Diamond\varphi \Rightarrow \Delta}\ (\Diamond\text{L'})$$

That is why we modify row ($\mathsf{A}_p^{\mathsf{CK}}16$) from Fig. 3 to the following row ( ):

($\mathsf{A}_p^{\mathsf{WK}}16$)	$\Gamma, \Diamond\delta \Rightarrow \Delta$	$\Box(\mathsf{E}_p(\Box^{-1}\Gamma, \delta) \to \mathsf{A}_p(\Box^{-1}\Gamma, \delta \Rightarrow \Diamond^{-1}\Delta))$

Now, $\mathsf{E}_p(\Gamma)$ and $\mathsf{A}_p(s)$ with respect to $\mathsf{G4WK}$ are defined as in Definition 5 with the modified table as described above. The analogues of Theorem 3 and Lemmas 3 and 4 hold for $\mathsf{G4WK}$ by the same proofs.

Theorem 5. *Sequent calculus* $\mathsf{G4WK}$ *has the uniform interpolation property.*

Proof. (, ,) Compared to the proof of uniform interpolation for $\mathsf{G4CK}$ (Theorem 4), a few modifications are needed, mostly in those places where ($\mathsf{A}_p^{\mathsf{CK}}16$) plays a role. The proof of the *implication* property (2a) does not change, and for (2b) the case for ($\mathsf{A}_p^{\mathsf{WK}}16$) works via the same proof as for ($\mathsf{A}_p^{\mathsf{CK}}16$) in Theorem 4, where we replace each occurrence of $\Diamond\delta_2$ by Δ and each δ_2 by $\Diamond^{-1}\Delta$. For the *uniformity* property, we check the case for (◇L') which comes with two subcases. They show similarities to the proof for (◇L) of Theorem 4:
Case (◇L'):
Subcase: The principal formula is in the p-free context Π, i.e., $\Pi = \Pi_0, \Diamond\delta'$.

(a) The conclusion of the rule is of the form $\Pi_0, \Diamond\delta', \Gamma \Rightarrow \Delta'$ where Δ' is p-free. By induction we have $\vdash_{\mathsf{G4WK}} \Box^{-1}\Pi_0, \delta', \mathsf{E}_p(\Box^{-1}\Gamma) \Rightarrow \Diamond^{-1}\Delta'$. We have:

$$\dfrac{\dfrac{\overset{\text{(IH)}}{\Box^{-1}\Pi_0, \delta', \mathsf{E}_p(\Box^{-1}\Gamma) \Rightarrow \Diamond^{-1}\Delta'}}{\Pi_0, \Diamond\delta', \Box\mathsf{E}_p(\Box^{-1}\Gamma) \Rightarrow \Delta'}\,(\Diamond\text{L'})}{\Pi_0, \Diamond\delta', \mathsf{E}_p(\Gamma) \Rightarrow \Delta'}\,\text{Lemma 4}$$

(b) The conclusion of the rule is of the form $\Pi_0, \Diamond\delta', \Gamma \Rightarrow \Delta$. We now consider two possibilities: $\Delta = \{\Diamond\psi\}$ for some ψ or not. If not, then $\Diamond^{-1}\Delta = \emptyset$ and $\mathsf{A}_p(\Gamma \Rightarrow \Delta)$ is not a $\Diamond$-formula by inspection of its construction. This means that we can reason as follows:

$$\dfrac{\dfrac{\overset{\text{(IH)}}{\Box^{-1}\Pi_0, \delta', \mathsf{E}_p(\Box^{-1}\Gamma) \Rightarrow}}{\Pi_0, \Diamond\delta', \Box\mathsf{E}_p(\Box^{-1}\Gamma) \Rightarrow \mathsf{A}_p(\Gamma \Rightarrow \Delta)}\,(\Diamond\text{L'})}{\Pi_0, \Diamond\delta', \mathsf{E}_p(\Gamma) \Rightarrow \mathsf{A}_p(\Gamma \Rightarrow \Delta)}\,\text{Lemma 4}$$

If $\Delta = \{\Diamond\psi\}$, then the same derivation applies as case (b) for $(\Diamond\text{L})$ in the proof of Theorem 4.

Subcase: The principal formula is in Γ, i.e., $\Gamma = \Gamma_0, \Diamond\delta_1$. This subcase is analogous to the previous subcase where instead of Lemma 4 we use Lemma 3 with $(\mathsf{E}_p^{\mathsf{WK}}11)$ and instead of using $(\mathsf{A}_p^{\mathsf{WK}}18)$ we apply $(\mathsf{A}_p^{\mathsf{WK}}16)$. ■

Corollary 2 ( ). *Logic* WK *has the uniform interpolation property.*

5 Conclusion

We have established the uniform interpolation property for constructive modal logics CK and WK, the first positive result of this kind among intuitionistic modal logics with independent $\Box$ and $\Diamond$. Our proof is *constructive* where we extend Pitts' proof technique to constructive $\Diamond$ by exploiting terminating sequent calculi for CK and WK. Our results have been formalised in Rocq which could contribute to the uniform interpolation calculator available in [26,27].

Our construction aims at showing that uniform interpolants *can* be constructed. Our approach is based on Pitts' naive construction, which can be simplified and optimised, as shown for the case of IL in [28]. Such simplifications, optimisations and hence more efficient algorithms, would be welcome in modal logics. A recent overview on complexity bounds on (uniform) interpolants in classical modal logics is provided in [17].

Another interesting direction for future research is to study the definability of so-called *bisimulation quantifiers*. These are interpreted over relational semantics and closely linked to uniform interpolants [1,50]. It would be interesting to see whether these are also definable in the birelational models of intuitionistic modal logics. Birelational semantics have been developed for CK [41] and WK [52] and a general study of such semantics is provided and formalised in Rocq in [34].

Finally, Pitts' technique received attention in the recent field of *universal proof theory* in which the uniform interpolation is linked to nicely behaved proof systems by inspecting the form of inference rules. So-called negative results have been established in the realm of intermediate logics stating that many of such logics cannot have a *semi-analytic* sequent calculus [38]. Modal logics are also treated in this context [3,4,36], but the negative results are in our opinion quite restrictive in the modal setting because only a few $\Box$-only rules are treated and a general format of 'modal rule' is not provided. We hope to provide a first step to widen the scope of universal proof theory by studying both $\Box$ and $\Diamond$.

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